Using qualitative subjective information in the form of "soft" inequalities to solve some ill-posed inverse problems

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Let's consider an inverse problem of estimating a latent "signal" $f \in \mathcal{R}_n$ (a vector of n-dimensional Euclidean space $\mathcal{R}_n$) from an "observation" $\xi \in \mathcal{R}_m$ (a vector of m-dimensional Euclidean space $\mathcal{R}_m$) that are connected as follows: $\xi = Af + \nu$, where A is a linear operator $\mathcal{R}_n \to \mathcal{R}_m$, and ν is a random noise with values in $\mathcal{R}_m$. According to the least squares method (LSM), an estimation of the signal f can be found as

$$\arg\min_g \|Ag - \xi\|^2 = A^+\xi, \tag{1}$$

where A^+ is a pseudoinverse of A.

It is known, that (1) is an ill-posed problem — if the operator A has "small" or zero singular values and is known approximately, i. e., an operator $A' \approx A$ is given instead of A, then the estimation $A'^+\xi$ can be very different from $A^+\xi$. In some cases, the difference can be arbitrary large even if A' is very close to A: $\|A'^+\xi - A^+\xi\| \xrightarrow[A' \to A]{} \infty$, see [1].

To get over this problem, we suggest to find an estimation $\hat{f}$ in the following form:

$$\hat{f} = \hat{f}^{(1)} + \hat{f}^{(2)}, \quad \hat{f}^{(1)} \in \mathcal{L} \triangleq \mathcal{L}(v_1, \ldots, v_k), \; \hat{f}^{(2)} \in \mathcal{L}^\perp, \tag{2}$$

where $\mathcal{L}$ is the linear span of the singular vectors $v_1, \ldots, v_k$ of A' corresponding to its "non-small" singular values. $\hat{f}^{(1)}$ is calculated as usual: $\hat{f}^{(1)} = A'^+_{\mathcal{L}}\xi$, where $A'_{\mathcal{L}}$ is the restriction of A' to $\mathcal{L}$.

To calculate $\hat{f}^{(2)}$, we suggest to use qualitative subjective information about likely monotonicity or constancy of the signal f (and, therefore, the estimation $\hat{f}$) in the form of "soft" inequalities between some of their coordinates f_i (and $\hat{f}_i$). For example, likely monotonicity of $\hat{f}$ can be expressed as $\ldots \lesssim f_{i-1} \lesssim f_i \lesssim f_{i+1} \lesssim \ldots$.

A notation "$x \lesssim y$" can be read as "most likely $x \leq y$" and can be modeled as an ill-known subset of $\mathbb{R} \times \mathbb{R}$ in the context of Pyt'ev possibility theory [2, 3]. In the work, the properties of the estimation (2) are studied and discussed.

References

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